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# AN ANALYTICAL AND NUMERICAL METHOD FOR SOLVING LINEAR AND NONLINEAR VIBRATION PROBLEMS

L. DAI and M. C. SINGH
Dept. Mechanical Engineering, The University of Calgary, 2500 University Drive N.W., Calgary, Alberta, Canada T2N 1N4

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**Abstract**—A direct analytical and numerical method independent of the existing classical methods for solving linear and nonlinear vibration problems is given with the introduction of a piecewise constant argument $[Nt]/N$. A new numerical method which produces sufficiently accurate results with good convergence is introduced. Development of the formulae for numerical calculations is based on the original governing differential equations. © 1997 Elsevier Science Ltd.

## 1. INTRODUCTION

The retarded and advanced functional differential equations with piecewise constant argument $[t]$ or $(t \pm n[t])$ have attracted a great deal of attention from the research workers (Busenberg *et al.*, 1982; Cooke *et al.*, 1984, 1987; Aftabizadeh *et al.*, 1985) in this area since the early 1980s. The first-order differential equations with a piecewise constant argument have been studied from the mathematical point of view of proofs of existence and uniqueness of solutions and oscillatory properties. Dai and Singh (1994) recently presented the solutions of several second-order differential equations representing motions of a spring-mass system disturbed by a piecewise constant force in the form of $f([t])$ or $f(x([t]))$. The oscillatory, nonoscillatory and periodic properties of motion of the spring-mass system were investigated. In this paper, the authors introduce a piecewise constant argument $[Nt]/N$ such that the differential equations studied by Busenberg *et al.* (1982), Cooke *et al.* (1984, 1987), Aftabizadeh *et al.* (1985), Dai and Singh (1994) with piecewise constant argument $[t]$ become special cases of the corresponding differential equations with the argument $[Nt]/N$ when $N = 1$. It is demonstrated in this paper that the argument $[Nt]/N$ tends to $t$ when $N$ approaches infinity in such a way that the governing differential equations with piecewise constant argument $[Nt]/N$ tend to become the differential equations with continuous argument $t$. With the introduction of the piecewise constant argument, the second-order differential equations which govern the vibration problems of an oscillatory system, can be directly solved through a piecewise-constant procedure.

In order to solve a second-order differential equation by a classical analytical method such as Euler's method discussed by Lancaster (1966) and Weaver *et al.* (1990), a form of the sought solution is assumed in advance and the complete solution of the differential equation depends on the assumed form of the solution and the characteristic equation produced by the assumed solution. In contrast with the classical methods, the approach presented in this paper generates solutions directly from the original differential equations, which govern the corresponding vibration problems, without any initial assumption for the form of the solutions.

It is shown in this paper that a function $f([Nt]/N)$ with the argument $[Nt]/N$ is a good approximation to the given continuous function $f(t)$ with argument $t$, if the parameter $N$ is sufficiently large. This makes the procedure of solving linear and nonlinear vibration problems an efficient numerical method. An important distinction between the present method and existing numerical methods is that the solutions given by the existing numerical methods are discrete, whereas the solutions and their first derivatives produced by the present method are continuous everywhere along the entire time range considered. Since the function $f([Nt]/N)$ is piecewise constant and $[Nt]$ is an integer, the recurrence relation used in numerical calculation can be easily derived and the numerical calculation can be conveniently carried out on a computer.

According to the convention, the mathematical symbol $[\cdot]$ used in this paper represents the greatest-integer function (Busenberg *et al.*, 1982; Cooke *et al.*, 1984, 1987; Aftabizadeh *et al.*, 1985). Assume $n$ is an integer and $y$ is a value in the range $n \leqslant y < n+1$, then $[y] = n$. Thus $[y]$ keeps a constant value $n$ as $y$ varies in between $n \leqslant y < n+1$. If $[y]$ is further employed as an argument, when $y$ increases continuously, the corresponding value of $[y]$ is piecewise constant.

## 2. PIECEWISE-CONSTANT METHOD IN SOLVING THE GOVERNING EQUATIONS FOR VIBRATION PROBLEMS

Vibratory motion associated with simple and complicated dynamic systems in engineering applications can often be simplified to the motion of a spring-mass system with one degree of freedom described by an equation of the form

$$m\ddot{x}+c\dot{x}+kx = f(t), \quad t > 0 \tag{1}$$

where the time dependent function $f(t)$ is known and expresses the continuous external force acting on the spring-mass system.

In order to represent a continuous system described in eqn (1) by a dynamic system with piecewise constant argument, the following theorem is introduced.

*Suppose an argument $[Nt]$ is the greatest-integer function of the product of time $t$ and a parameter $N$, where $N$ is a real positive integer (for the sake of convenience, though $N$ is not necessarily an integer), then, ratio $[Nt]/N$ approaches $t$ as $N$ goes to infinity.*

The theorem can be proved as follows. For a given $\varepsilon > 0$, there exists a number $N_0$ such that $1/N_0 < \varepsilon$. Since $[Nt]/N-t = ([Nt]-Nt)/N$ and $-1 \leqslant [Nt]-Nt \leqslant 0$, hence, $|[Nt]/N-t| \leqslant N$. Accordingly for any $\varepsilon > 0$, take $N_0 \geqslant [1/\varepsilon]$, when $N \geqslant N_0$. It follows that

$$\left|\frac{[Nt]}{N}-t\right| \leqslant \frac{1}{N} \leqslant \frac{1}{N_0} < \varepsilon. \tag{2}$$

By the arbitrariness of $\varepsilon$, a result of limiting case can be given as follows:

$$\lim_{N\to\infty} \frac{[Nt]}{N} = t. \tag{3}$$

The theorem is thus proved.

The above theorem and the conclusion given in eqn (3) is extremely important since most of the derivations in the next section will depend on them. With the theorem, eqn (1) can be solved by replacing the variable $t$ of the continuous function $f(t)$ in eqn (1) with the piecewise constant variable $[Nt]/N$ such that a piecewise constant system is obtained with the governing equation:

$$m\ddot{X}+c\dot{X}+kX = g\left(\frac{[Nt]}{N}\right), \quad [Nt]/N \leqslant t < ([Nt]+1)/N. \tag{4}$$

For the purpose of distinction between the continuous system governed by eqn (1) and the piecewise constant system indicated above, and for convenience of further discussion, $X$ here is designated for the displacement of the piecewise constant system governed by eqn (4). The forcing function $g([Nt]/N)$ in eqn (4) stands for the piecewise constant force acting on the system. $g([Nt]/N)$ is discontinuous on the entire range of $t \in [0, \infty)$, and only varies its value at the moment $t = i/N$, $i = 1,2,3,\ldots,[Nt]$. The value of $g([Nt]/N)$ is constant in the interval $i/N \leqslant t < (i+1)/N$ and $g([Nt]/N)$ has the same values as $f(t)$ at $t = i/N$, $i = 1, 2, 3, \ldots, [Nt]$, i.e., $g(i/N) = f(i/N)$. This implies that the length of the interval is $1/N$. The length of the interval is less than a unit of time if the parameter $N$ is taken a value greater than one. Theoretically, according to the theorem, the length of the interval approaches zero as $N$ tends to infinity.

Based on the discussion above, eqn (4) represents a simplified continuous dynamic system of linear vibration within a time segment of length $1/N$. When the parameter $N$ is made large enough, the time interval is made correspondingly as small as desired, the motion governed by eqn (4) can be approximately considered as the vibration governed by eqn (1). It will be shown in the following section that the solutions of eqns (1) and (4) are identical if $N$ tends to infinity. The procedure of representing a continuous function in eqn (1) by a piecewise constant function over $i/N \leqslant t < (i+1)/N$ is termed as "Piecewise Constant Procedure" hereafter.

A function $X(t)$ is a solution of eqn (4) only if the following conditions are satisfied.

(i) $X(t)$ and its derivative $\dot{X}(t)$ are continuous over $t \in [0, \infty)$;
(ii) $\ddot{X}(t)$ exists at each point of time with the possible exception at the points $i/N$ where the left-sided derivatives exist;
(iii) on each time interval $i/N \leqslant t < (i+1)/N$, $X(t)$ satisfies eqn (4);
(iv) the general solution $X(t)$ is a combination of any particular solution of the inhomogeneous eqn (4) and the solution of the corresponding homogeneous equation.

## 3. ANALYTICAL SOLUTIONS THROUGH PIECEWISE CONSTANT PROCEDURE

Based on the existence and uniqueness of the solution $X(t)$ on $[0, \infty)$ and $X_i(t)$ for the interval $i/N \leqslant t < (i+1)/N$, the complete solution for $x(t)$ in eqn (1) can be derived by employing the piecewise constant argument $[Nt]/N$. For the purpose of clarification, a free vibration without damping, i.e., $c = 0$ and $f(t) = 0$ in eqn (1), will first be solved by the piecewise constant procedure.

### (a) *Free vibration without damping*

Consider an equation of motion of an undamped free vibratory system:

$$\ddot{x}(t) + \omega^2 x(t) = 0 \tag{5}$$

where $\omega^2 = k/m$.

The initial conditions are given by

$$x(t = 0) = d_0 \quad \text{and} \quad \dot{x}(t = 0) = v_0. \tag{6}$$

The term $\omega^2 x(t)$ in eqn (5) can be represented by a piecewise constant function over an arbitrary time interval $i/N \leqslant t < (i+1)/N$ and the corresponding equation of motion is expressible in the form

$$\ddot{X}_i(t) + \omega^2 X_i\left(\frac{i}{N}\right) = 0 \tag{7}$$

for any time interval $i/N \leqslant t < (i+1)/N$, $i = 0, 1, 2, 3, \ldots, [Nt]$. Equation (7) is now a simplified linear differential equation. In eqn (7), the subscript $i$ represents the $i$th time interval from the origin $t = 0$. The term $\omega^2 X_i(i/N)$ in the equation is constant over the time segment $i/N \leqslant t < (i+1)/N$. Because of this, eqn (7) can be directly integrated over this interval to obtain a general solution.

$$X_i(t) = \left[1-\frac{\omega^2}{2}\left(t-\frac{i}{N}\right)^2\right]d_i+\left(t-\frac{i}{N}\right)v_i \tag{8}$$

where $d_i$ and $v_i$ are displacement and velocity respectively at $t = i/N$. Similarly, denoting $X_{i-1}$ as the solution on the interval $(i-1)/N \leqslant t < i/N$, the integration of the corresponding differential equation gives

$$X_{i-1}(t) = \left[1-\frac{\omega^2}{2}\left(t-\frac{i-1}{N}\right)^2\right]d_{i-1}+\left(t-\frac{i-1}{N}\right)v_{i-1} \tag{9}$$

where

$$d_{i-1} = X_{i-1}\left(\frac{i-1}{N}\right) \quad \text{and} \quad v_{i-1} = \dot{X}_{i-1}\left(\frac{i-1}{N}\right). \tag{10}$$

As the motion of the system is continuous, the displacement $X(t)$ and velocity $\dot{X}(t)$ must be continuous on $t \in [0, \infty)$. Because of the continuity of $X$ and $\dot{X}$, the following conditions must be satisfied:

$$X_i\left(\frac{i}{N}\right) = X_{i-1}\left(\frac{i}{N}\right) \quad \text{and} \quad \dot{X}_i\left(\frac{i}{N}\right) = \dot{X}_{i-1}\left(\frac{i}{N}\right). \tag{11}$$

With the above conditions of continuity, a recurrence relation is obtained by substituting eqns (8) and (9) in eqn (11) as

$$\begin{bmatrix} d_i \\ v_i \end{bmatrix} = \begin{bmatrix} 1-\omega^2/2N^2 & 1/N \\ -\omega^2/N & 1 \end{bmatrix}\begin{bmatrix} d_{i-1} \\ v_{i-1} \end{bmatrix}. \tag{12}$$

As a consequence of an iterative procedure, $d_i$ and $v_i$ can be connected to the initial displacement $d_0$ and initial velocity $v_0$ in the form

$$\begin{bmatrix} d_i \\ v_i \end{bmatrix} = \begin{bmatrix} 1-\omega^2/2N^2 & 1/N \\ -\omega^2/N & 1 \end{bmatrix}^i \begin{bmatrix} d_0 \\ v_0 \end{bmatrix}. \tag{13}$$

It is clear that the displacement and velocity of the system at any given point of time $i/N$ can be calculated by using eqn (13).

Considering that the $i$th time interval is arbitrarily chosen and $[Nt]$ can be any integer number represented by $i$, the complete solution $X(t)$ may be written on the basis of eqns (8) and (13) as

$$X(t) = \left[1-\frac{\omega^2}{2}\left(t-\frac{[Nt]}{N}\right)^2, \ t-\frac{[Nt]}{N}\right]\begin{bmatrix} 1-\omega^2/2N^2 & 1/N \\ -\omega^2/N & 1 \end{bmatrix}^{[Nt]}\begin{bmatrix} d_0 \\ v_0 \end{bmatrix}. \tag{14}$$

Note that the solution is valid on the entire range, $t \geqslant 0$, and it is continuous on any interval $i/N \leqslant t < (i+1)/N$ as well as the range $[0, \infty)$. For any time $t$ there is a value of $X(t)$ corresponding to it. A limit of eqn (14) can be further taken as $N$ approaches infinity.

Let **Q** represent the square matrix in eqn (14)

$$\mathbf{Q} = \begin{bmatrix} 1-\dfrac{\omega^2}{2N^2} & \dfrac{1}{N} \\ -\dfrac{\omega^2}{N} & 1 \end{bmatrix}. \tag{15}$$

According to Lancaster (1966), **Q** can be written in the following form:

$$\mathbf{Q} = \mathbf{EDE}^{-1} \tag{16}$$

where **E** is a set of linearly independent eigenvectors of **Q** and **D** a diagonal matrix of the eigenvalues of **Q**. Designating $\phi_i$ as eigenvalues of the matrix **Q** and noting that $\mathbf{EE}^{-1}$ is equivalent to an identity matrix, by Lancaster (1966), the exponential matrix of **Q** may now be written as

$$\mathbf{Q}^{[Nt]} = \mathbf{ED}^{[Nt]}\mathbf{E}^{-1} = \mathbf{E}\begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}^{[Nt]}\mathbf{E}^{-1} = \mathbf{E}\begin{bmatrix} \phi_1^{[Nt]} & 0 \\ 0 & \phi_2^{[Nt]} \end{bmatrix}\mathbf{E}^{-1} \tag{17}$$

where the eigenvalues $\phi_1$ and $\phi_2$ are determined from eqn (15) as

$$\phi_1 = 1-\frac{\omega^2}{4N^2}+\frac{\omega^2}{4N^2}\left(1-\frac{16N^2}{\omega^2}\right)^{1/2} \tag{18}$$

$$\phi_2 = 1-\frac{\omega^2}{4N^2}-\frac{\omega^2}{4N^2}\left(1-\frac{16N^2}{\omega^2}\right)^{1/2}. \tag{19}$$

Corresponding to these eigenvalues, **E** and the inverse of **E** can be calculated as follows:

$$\mathbf{E} = \begin{bmatrix} \dfrac{1}{4N}\left[1-\left(1-\dfrac{16N^2}{\omega^2}\right)^{1/2}\right] & \dfrac{1}{4N}\left[1-\left(1-\dfrac{16N^2}{\omega^2}\right)^{1/2}\right] \\ 1 & 1 \end{bmatrix} \tag{20}$$

$$\mathbf{E}^{-1} = \begin{bmatrix} -\dfrac{2N}{\left(1-\dfrac{16N^2}{\omega^2}\right)^{1/2}} & \dfrac{1}{2}+\dfrac{1}{2\left(1-\dfrac{16N^2}{\omega^2}\right)^{1/2}} \\ \dfrac{2N}{\left(1-\dfrac{16N^2}{\omega^2}\right)^{1/2}} & \dfrac{1}{2}-\dfrac{1}{2\left(1-\dfrac{16N^2}{\omega^2}\right)^{1/2}} \end{bmatrix}. \tag{21}$$

Taking these facts into consideration, the general solution (14) can be given in the following form:

$$X(t) = \frac{1}{a_{11}-a_{12}}[b_{11} \quad b_{12}]\mathbf{A}\begin{bmatrix} d_0 \\ v_0 \end{bmatrix} \tag{22}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11}\phi_1^{[Nt]} - a_{12}\phi_2^{[Nt]} & -a_{11}a_{12}\phi_1^{[Nt]} + a_{11}a_{12}\phi_2^{[Nt]} \\ \phi_1^{[Nt]} - \phi_2^{[Nt]} & -a_{12}\phi_1^{[Nt]} + a_{11}\phi_2^{[Nt]} \end{bmatrix} \tag{23}$$

and

$$a_{11} = \frac{1}{4N} - \left(\frac{1}{16N^2} - \frac{1}{\omega^2}\right)^{1/2}, \quad a_{12} = \frac{1}{4N} + \left(\frac{1}{16N^2} - \frac{1}{\omega^2}\right)^{1/2} \tag{24a,b}$$

$$b_{11} = 1 - \frac{\omega^2}{2}\left(t - \frac{[Nt]}{N}\right)^2, \quad b_{12} = t - \frac{[Nt]}{N}. \tag{24c,d}$$

In calculating $\lim_{N\to\infty} X(t)$ with the help of l'Hopital's rule, the following useful results can be obtained from eqns (18) and (19):

$$\lim_{N\to\infty} \phi_1^{[Nt]} = e^{it\omega} \tag{25}$$

$$\lim_{N\to\infty} \phi_2^{[Nt]} = e^{it\omega}. \tag{26}$$

Further, on the basis of eqns (25) and (26),

$$\lim_{N\to\infty} \frac{-a_{11}\phi_1^{[Nt]} - a_{12}\phi_2^{[Nt]}}{a_{11} - a_{12}} = \frac{e^{i\omega t} + e^{-i\omega t}}{2} = \cos(\omega t) \tag{27}$$

and

$$\lim_{N\to\infty} \frac{-a_{11}a_{12}\phi_1^{[Nt]} + a_{11}a_{12}\phi_2^{[Nt]}}{a_{11} - a_{12}} = \frac{1}{\omega}\sin(\omega t). \tag{28}$$

Substituting eqns (25), (26), (27) and (28) in eqn (22), the final result is obtained as

$$\lim_{N\to\infty} X(t) = \begin{bmatrix} \cos(\omega t) & \frac{1}{\omega}\sin(\omega t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d_0 \\ v_0 \end{bmatrix} \tag{29}$$

which is exactly the same solution as the complete classical analytical solution demonstrated by Lancaster (1966) for an undamped free vibration system governed by eqn (5), i.e.,

$$\lim_{N\to\infty} X(t) = x(t) = d_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t. \tag{30}$$

The difference between the solution of the continuous system governed by eqn (5) and the piecewise constant system expressed in eqn (7) vanishes in the limiting case as the parameter $N$ in eqn (7) approaches infinity. Equation (7) may therefore be considered as a more general equation of motion covering both a piecewise constant system and in the limit a continuous system. It may be noted in eqn (7) that as $N$ takes on a finite value, the system is piecewise constant; but when $N$ tends to infinity, the corresponding system become continuous representing a linear vibration. It may also be seen from the above discussion that the linear vibration problem governed by eqn (5) is analytically solved by an approach independent of the classical analytical methods discussed by Weaver *et al.* (1990).

(*b*) *Forced vibration without damping*

With a sinusoidal external force, the equation of motion for an undamped linear spring-mass system is

$$\ddot{x}+\omega^2 x = F\cos\Omega t \tag{31}$$

where $F$ refers to the amplitude of the external force and $\Omega$ is its angular frequency.

Since the solution of the homogeneous part of eqn (31), $\ddot{x}+\omega^2 x = 0$, has been obtained previously in part (a), the equation of motion is expressed in the following piecewise constant form:

$$\ddot{X}_i+\omega^2 X_i = F\cos\left(\Omega\frac{i}{N}\right) \tag{32}$$

which is valid on an $i$th interval of $i/N \leqslant t < (i+1)/N$.

By the same procedure as discussed in part (a), with identical initial and continuity conditions, the general solution of eqn (32) can be expressed on the basis of the solution (30) as

$$X(t) = \Psi_1+\Psi_2+\Psi_3 \tag{33}$$

where

$$\Psi_1 = \left[\cos\left[\omega\left(t-\frac{[Nt]}{N}\right)\right]\frac{1}{\omega}\sin\left[\omega\left(t-\frac{[Nt]}{N}\right)\right]\right]\mathbf{T}^{[Nt]}\begin{bmatrix} d_0 \\ v_0 \end{bmatrix} \tag{34}$$

$$\Psi_2 = \left[\cos\left[\omega\left(t-\frac{[Nt]}{N}\right)\right]\frac{1}{\omega}\sin\left[\omega\left(t-\frac{[Nt]}{N}\right)\right]\right] \cdot\sum_{r=1}^{[Nt]}\left\{\frac{F}{\omega^2}\mathbf{T}^{r-1}\begin{bmatrix} \cos\frac{\omega}{N}-1 \\ -\omega\sin\frac{\omega}{N} \end{bmatrix}\cos\left[\Omega\left(\frac{[Nt]-r}{N}\right)\right]\right\} \tag{35}$$

$$\Psi_3 = \frac{F}{\omega^2}\cos\left(\Omega\frac{[Nt]}{N}\right)\left[1-\cos\left(\omega t-\omega\frac{[Nt]}{N}\right)\right]. \tag{36}$$

In the above equations, $\mathbf{T}$ is a square matrix of the form

$$\mathbf{T} = \begin{bmatrix} \cos\frac{\omega}{N} & \frac{1}{\omega}\sin\frac{\omega}{N} \\ -\omega\sin\frac{\omega}{N} & \cos\frac{\omega}{N} \end{bmatrix} \tag{37}$$

with the eigenvalues

$$\phi_1 = e^{i\omega/N} \quad \text{and} \quad \phi_2 = e^{-i\omega/N}. \tag{38}$$

The matrix $\mathbf{T}$ with power $[Nt]$, similar in form to that in eqn (17), may then be represented by the following expression with the corresponding eigenvectors of the square matrix $\mathbf{T}$:

$$\mathbf{T}^{[Nt]} = \frac{1}{2}\begin{bmatrix} -\frac{i}{\omega} & \frac{i}{\omega} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{i\omega([Nt]/N)} & 0 \\ 0 & e^{-i\omega([Nt]/N)} \end{bmatrix}\begin{bmatrix} -\frac{\omega}{i} & 1 \\ \frac{\omega}{i} & 1 \end{bmatrix}. \tag{39}$$

When the parameter $N$ approaches infinity, from eqn (34) it can be shown that

$$\lim_{N\to\infty} \Psi_1 = \begin{bmatrix} \cos\omega t & 0 \\ 0 & \frac{1}{\omega}\sin\omega t \end{bmatrix}\begin{bmatrix} d_0 \\ v_0 \end{bmatrix}. \tag{40}$$

This is a free vibration in exactly the same form as the classical solution presented by Weaver *et al.* (1990) for the homogeneous part of the continuous governing eqn (31).

Noting that the sum after the symbol $\Sigma_{r=1}^{[Nt]}$ in eqn (35) is with respect to the argument $r$, the summation of all the terms with argument $r$ can be carried out to give the following result:

$$\mathbf{M} = \sum_{r=1}^{[Nt]} \frac{1}{2}\begin{bmatrix} -\frac{i}{\omega} & \frac{i}{\omega} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{i\omega(r-1/N)} & 0 \\ 0 & e^{-i\omega(r-1/N)} \end{bmatrix}\begin{bmatrix} -\frac{\omega}{i} & 1 \\ \frac{\omega}{i} & 1 \end{bmatrix}\cos\left[\Omega\left(\frac{[Nt]-r}{N}\right)\right]$$

$$= \frac{1}{2}\begin{bmatrix} -\frac{i}{\omega} & \frac{i}{\omega} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}\begin{bmatrix} -\frac{\omega}{i} & 1 \\ \frac{\omega}{i} & 1 \end{bmatrix} \tag{41}$$

where

$$L_1 = \frac{1}{2}\left[\frac{e^{i/N([Nt]\Omega-\Omega)} - e^{i/N([Nt]\omega-\Omega)}}{1-e^{i/N(\omega-\Omega)}} + \frac{e^{-i/N([Nt]\Omega-\Omega)} - e^{i/N([Nt]\omega+\Omega)}}{1-e^{i/N(\omega+\Omega)}}\right] \tag{42}$$

$$L_2 = \frac{1}{2}\left[\frac{e^{i/N([Nt]\Omega-\Omega)} - e^{-i/N([Nt]\omega+\Omega)}}{1-e^{-i/N(\omega+\Omega)}} + \frac{e^{-i/N([Nt]\Omega-\Omega)} - e^{-i/N([Nt]\omega-\Omega)}}{1-e^{i/N(\omega-\Omega)}}\right]. \tag{43}$$

Making use of the results given by eqns (3) and (41) together with l'Hopital's rule, the limit of the second term in eqn (33) may be obtained to have the following simple form:

$$\lim_{N\to\infty} \Psi_2 = \frac{F}{\omega^2-\Omega^2}(\cos\Omega t - \cos\omega t). \tag{44}$$

The limit of the last term,

$$\lim_{N\to\infty} \Psi_3 = \lim_{N\to\infty} \frac{F}{\omega^2}\cos\left(\Omega\frac{[Nt]}{N}\right)\left[1-\cos\left(\omega t - \omega\frac{[Nt]}{N}\right)\right] = 0. \tag{45}$$

Substituting the results of eqns (40), (44) and (45) into eqn (33) and taking a limit as $N \to \infty$, the limit of the solution to eqn (32) is now obtained in the following form:

$$\lim_{N\to\infty} X(t) = x(t) = \begin{bmatrix} \cos\omega t & 0 \\ 0 & \frac{1}{\omega}\sin\omega t \end{bmatrix} \begin{bmatrix} d_0 \\ v_0 \end{bmatrix} + \frac{F}{\omega^2-\Omega^2}(\cos\Omega t - \cos\omega t). \quad (46)$$

This is identical to the solution of eqn (31) obtained by using the classical analytical methods demonstrated by Weaver *et al.* (1990).

(*c*) *Free vibrations with damping*

From eqn (1), the equation of motion for free vibration with viscous damping is expressible as

$$\ddot{x} + 2n\dot{x} + \omega^2 x = 0 \quad (47)$$

where $2n = c/m$, and the corresponding initial conditions are as shown in eqn (6). By a piecewise constant procedure, eqn (47) can be expressed in the time interval $i/N \leqslant t < (i+1)/N$ as

$$\ddot{X}_i + X_i = -2n\dot{X}_i\left(\frac{i}{N}\right). \quad (48)$$

Considering that the term $2n\dot{X}_i(i/N)$ is constant over $i/N \leqslant t < (i+1)/N$, the results obtained in part (a) may be employed to yield the homogeneous part of the solution to eqn (48). On the basis of solution (30), and taking into account the conditions given by eqn (11), the general solution of eqn (48) is expressible in the form

$$X_i(t) = d_i \cos\left[\omega\left(t-\frac{i}{N}\right)\right] + \frac{v_i}{\omega}\sin\left[\omega\left(t-\frac{i}{N}\right)\right] - v_i\frac{2n}{\omega^2}\left\{1-\cos\left[\omega\left(t-\frac{j}{N}\right)\right]\right\}. \quad (49)$$

By the same procedure as for obtaining eqn (14), and with the continuity conditions in eqn (11), the complete solution $X(t)$ of eqn (48) has been derived as

$$X(t) = \left[\cos\left[\omega\left(t-\frac{[Nt]}{N}\right)\right]\frac{1}{\omega}\sin\left[\omega\left(t-\frac{[Nt]}{N}\right)\right] - \frac{2n}{\omega^2}\left\{1-\cos\left[\omega\left(t-\frac{[Nt]}{N}\right)\right]\right\}\right]\begin{bmatrix} d_i \\ v_i \end{bmatrix} \quad (50)$$

where

$$\begin{bmatrix} d_i \\ v_i \end{bmatrix} = \begin{bmatrix} \cos\frac{\omega}{N} & \frac{1}{\omega}\sin\frac{\omega}{N} - \frac{2n}{\omega^2}\left(1-\cos\frac{\omega}{N}\right) \\ -\omega\sin\frac{\omega}{N} & \cos\frac{\omega}{N} - \frac{2n}{\omega}\sin\frac{\omega}{N} \end{bmatrix}^{[Nt]} \begin{bmatrix} d_0 \\ v_0 \end{bmatrix}. \quad (51)$$

The procedure of taking a limit as $N$ approaches infinity has been clarified in parts (a) and (b). The principles in taking the limits are all the same, and the methods employed in obtaining the limits are fairly elementary. By the same procedure for finding the limits, the limit of the closed-form result indicated in eqn (50) as $N$ approaches infinity can be determined. The limit is obtained as

$$\lim_{N\to\infty} X(t) = x(t) = e^{-nt}\begin{bmatrix}\cos\xi t & \frac{1}{\xi}\sin\xi t\end{bmatrix}\begin{bmatrix}d_0\\ v_0\end{bmatrix}+e^{-nt}\frac{nd_0}{\xi}\sin\xi t \tag{52}$$

where $\xi = \sqrt{\omega^2-n^2}$ and $n < \omega$. This solution is of the same form as the classical analytical methods produced by Weaver *et al.* (1990) to eqn (47). The other two solutions for the cases $n > \omega$ and $n = \omega$ may also be derived in a similar manner as for deriving eqn (52). Again, the solutions obtained are identical to the corresponding solutions given by the classical analytical methods.

(*d*) *Forced vibrations with damping*

If a harmonic excitation is applied to the system governed by eqn (47), the corresponding vibratory motion may be described by

$$\ddot{x}+2n\dot{x}+\omega^2x = F\cos\Omega t \tag{53}$$

with initial conditions the same as in eqn (6). Since the homogeneous equation corresponding to eqn (53) has been solved previously, the governing equation may be expressed in a piecewise constant form over $i/N \leqslant t < (i+1)/N$ as

$$\ddot{X}_i+2n\dot{X}_i+\omega^2X_i = F\cos\left(\Omega\frac{i}{N}\right) \tag{54}$$

and, with the same procedure as employed in obtaining eqn (14), the complete solution $X(t)$ can be found for the case $n < \omega$ as

$$\begin{aligned} X(t) &= e^{-n(t-[Nt]/N)}\left[\cos\left(\xi t-\xi\frac{[Nt]}{N}\right)+\frac{n}{\xi}\sin\left(\xi t-\xi\frac{[Nt]}{N}\right)\ \frac{1}{\xi}\sin\left(\xi t-\xi\frac{[Nt]}{N}\right)\right]\mathbf{W} \\ &+\frac{F}{\omega^2}e^{-n(t-[Nt]/N)}\left[e^{n(t-[Nt]/N)}-\cos\left(\xi t-\xi\frac{[Nt]}{N}\right)-\frac{n}{\xi}\sin\left(\xi t-\xi\frac{[Nt]}{N}\right)\right]\cos\left(\Omega\frac{[Nt]}{N}\right) \end{aligned} \tag{55}$$

where the matrix $\mathbf{W}$ is

$$\mathbf{W} = e^{-n[Nt]/N}\mathbf{G}^{[Nt]}\begin{bmatrix}d_0\\ v_0\end{bmatrix}+\sum_{r=1}^{[Nt]} e^{-nr/N}\mathbf{G}^{r-1}\begin{bmatrix} e^{n/N}-\cos\frac{\xi}{N}-\frac{n}{\xi}\sin\frac{\xi}{N} \\ \left(\frac{n^2}{\xi}+\xi\right)\sin\frac{\xi}{N}\end{bmatrix}\frac{F}{\omega^2}\cos[\Omega([Nt]-r)] \tag{56}$$

in which the square matrix $\mathbf{G}$ has the form

$$\mathbf{G} = \begin{bmatrix} \cos\frac{\xi}{N}+\frac{n}{\xi}\sin\frac{\xi}{N} & \frac{1}{\xi}\sin\frac{\xi}{N} \\ -\left(\frac{n^2}{\xi}+\xi\right)\sin\frac{\xi}{N} & \cos\frac{\xi}{N}-\frac{n}{\xi}\sin\frac{\xi}{N}\end{bmatrix}. \tag{57}$$

The limit of $X(t)$ in eqn (55), as $N\to\infty$, can be found with the use of eqns (56) and (57) as

$$\lim_{N\to\infty} X(t) = x(t) = e^{-nt}\left[\cos\xi t+\frac{n}{\xi}\sin\xi t \quad \frac{1}{\xi}\sin\xi t\right]\begin{bmatrix} d_0 \\ v_0 \end{bmatrix}$$
$$+\frac{F}{(\omega^2-\Omega^2)^2+(2n\Omega)^2}\left[(\omega^2-\Omega^2)(\cos\Omega t-e^{-nt}\cos\xi t)\right.$$
$$\left.+2n\Omega\sin\Omega t-e^{-nt}(\omega^2+\Omega^2)\frac{n}{\xi}\sin\xi t\right]. \quad (58)$$

As it should be, the solution above is identical to the corresponding classical analytical solution $x(t)$ of eqn (53) presented by Weaver *et al.* (1990).

For vibration problems governed by eqn (1), solutions corresponding to free and forced vibrations with or without damping have been obtained by the piecewise constant procedure and all the solutions have been proved to converge to the corresponding classical solutions of eqn (1) as $N \to \infty$. Thus, it may be stated that, for any $t$, if

(i) $X(t)$ satisfies eqns (4) and (11) in $[i/N,(i+1)/N]$,
(ii) $d_i = \lim_{t\to(i/N)^-} X(t)$, and $v_i = \lim_{t\to(i/N)^-} \dot{X}(t)$,

as $N \to \infty$, then $X(t)$ in eqn (4) must converge to $x(t)$ which satisfies eqn (1) with the corresponding initial conditions.

## 4. NUMERICAL SOLUTIONS FOR VIBRATORY PROBLEMS

As was seen in Sections 2 and 3 above, with a sufficiently large $N$, solution $X$ of the piecewise constant system approximately represents the solution $x(t)$ of the corresponding continuous system in the same interval of time. When the parameter $N$ approaches infinity, the interval of time segment tends to zero, the solution of a dynamic system with a piecewise constant argument becomes the solution of the corresponding continuous system. This enables one to make use of the piecewise-constant method for the purpose of numerically solving the vibration problems. Due to the properties of the integer argument $[Nt]$ on the time range considered, the piecewise-constant method is practically convenient in applying on a computer for numerical solutions of differential equations in dynamics.

(1) *Numerical solution of a linear system*

A numerical solution of the governing eqn (5) for a linear undamped free system may first be considered. With the piecewise-constant eqn (7), the solution and recurrence relations corresponding to the $i$th time interval can be written as

$$X_i(t) = \left[1-\frac{\omega^2}{2}\left(t-\frac{i}{N}\right)^2\right]d_i+\left(t-\frac{i}{N}\right)v_i \quad (59)$$

$$\dot{X}_i(t) = -\omega^2\left(t-\frac{i}{N}\right)d_i+v_i \quad (60)$$

$$d_i = \left[1-\frac{\omega^2}{2N^2}\right]d_{i-1}+\frac{1}{N}v_{i-1} \quad (61)$$

$$v_i = -\left(\frac{\omega^2}{N}\right)d_{i-1}+v_{i-1}. \quad (62)$$

Once the initial conditions are given, the displacement $X_1$ and velocity $\dot{X}_1$ at the end of the first interval are readily available from the above equations. Through a step-by-step procedure, eqns (59), (60), (61) and (62) lead to a numerical solution of eqn (5). From eqns (59) and (60), it is evident that a solution of the piecewise-constant dynamical system and its first derivative are continuous in the interval, $i/N \leqslant t < i+1/N$.

It has been found in the process of the numerical computations that the solutions with recurrence relations given above are concise and manageable on computers. In addition, the solutions and the corresponding recurrence relations are easy to construct as shown in Section 3.

As an example, consider the forced vibration with damping governed by eqn (53). Based on eqn (55), the numerical solution for this system can be obtained by use of the following formulae:

$$X_i = e^{-n(t-i/N)}\left\{(d_i-\gamma_i)\cos\left[\xi\left(t-\frac{i}{N}\right)\right]+\frac{1}{\xi}[v_i+n(d_i-\gamma_i)]\sin\left[\xi\left(t-\frac{i}{N}\right)\right]\right\}+\gamma_i \quad (63)$$

$$\dot{X}_i = -ne^{-n(t-i/N)}\left\{(d_i-\gamma_i)\cos\left[\xi\left(t-\frac{i}{N}\right)\right]+\frac{1}{\xi}[v_i+n(d_i-\gamma_i)]\sin\left[\xi\left(t-\frac{i}{N}\right)\right]\right\}$$
$$+e^{-n(t-i/N)}\left\{-\xi(d_i-\gamma_i)\sin\left[\xi\left(t-\frac{i}{N}\right)\right]+[v_i+n(d_i-\gamma_i)]\cos\left[\xi\left(t-\frac{i}{N}\right)\right]\right\} \quad (64)$$

$$d_i = e^{-n/N}\left[(d_{i-1}-\gamma_{i-1})\cos\frac{\xi}{N}+\frac{1}{\xi}(v_{i-1}+nd_{i-1}-n\gamma_{i-1})\sin\frac{\xi}{N}\right]+\gamma_{i-1} \quad (65)$$

$$v_i = e^{-n/N}\left\{v_{i-1}\cos\frac{\xi}{N}-\left[(v_{i-1}+nd_{i-1}-n\gamma_{i-1})\frac{n}{\xi}+\xi(d_{i-1}-\gamma_{i-1})\right]\sin\frac{\xi}{N}\right\} \quad (66)$$

where

$$\gamma_i = \frac{F}{\omega^2}\cos\frac{i}{N}. \quad (67)$$

Starting with the initial conditions, the above relations are used in a computer program for obtaining solution in each time interval. The numerical results obtained by the application of the above equations are illustrated in Fig. 1. As shown in the figure, the error of the numerical results from the exact solution is already small when $N$ assumes a value of 10. The numerical results are getting closer and closer to the exact solution of the system as $N$ increases. The parameter $N$ obviously acts as a factor controlling the accuracy of the numerical solution. In order to have a numerical solution of high accuracy, one may simply choose a large enough value for $N$.

It is significant to mention here that the parameter $N$ is actually a factor in controlling the interval $i/N \leqslant t < i+1/N$, which is the step length in the numerical calculations by the piecewise-constant method. Although the solution given by the piecewise-constant method will, theoretically, tend to the exact solution as $N$ approaches infinity, for practical reasons, however, the step length cannot be taken to be too small or correspondingly, $N$ cannot be taken to be too large. As pointed by Boyce and DiPrima (1969), if the step length is too small, a large number of steps will be required to operate in a fixed time interval which may in turn cause a considerable increase of the round-off error. The piecewise-constant method is a single step method. To carry out the numerical calculations by the piecewise-constant method, Press's technique (Press *et al.*, 1986) has been employed for determining the proper step length.

It should be noted that the numerical solutions above are:

(i) derived directly from the corresponding second-order differential equations;
(ii) valid on the time interval $i/N \leqslant t < (i+1)/N$;

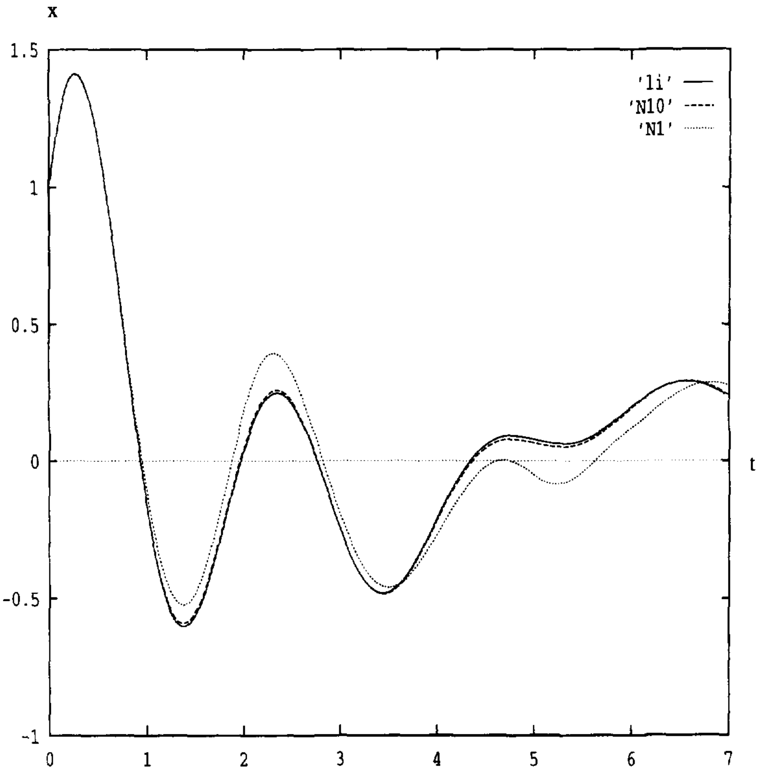


Fig. 1. Convergence of the solution of $\ddot{x}(t)+2n\dot{x}(t)+\omega^2 x([Nt]/N) = F\cos\Omega t$ to that of $\ddot{x}(t)+2n\dot{x}(t)+\omega^2 x(t) = F\cos\Omega t$ (solid line, 'li'). $n = 0.5$, $\omega = 3$, $F = 2$, and $\Omega = 1$. In the figure, '$N10$' : $N = 10$; '$N1$' : $N = 1$.

(iii) continuous everywhere on $i/N \leqslant t < (i+1)/N$ (i.e., in between $X_i$ and $X_{i+1}$). The continuity of the solutions is independent of the length of the interval which is controlled by the parameter $N$.

(2) *Numerical solutions for nonlinear systems*

As an important consequence, the present method can be used to solve nonlinear vibration problems by linearizing the nonlinear terms of the corresponding nonlinear differential equations through the piecewise constant procedure.

(*a*) *Duffing's equation.* Consider a nonlinear oscillation governed by the well known Duffing's equation with a linear and cubic stiffness:

$$\ddot{x}(t)+2\theta\dot{x}(t)+\omega^2 x(t)+\beta x^3(t) = 0. \tag{68}$$

This equation of motion may be used to model, for example, a spring-mass system having a hardening spring discussed by Lancaster (1966) or a buckled beam under harmonic excitation studied by Dowell *et al.* (1986). By the piecewise constant procedure, the above equation of motion is expressible as

$$\ddot{X}(t)+2\theta\dot{X}(t)+\omega^2 X(t)+\beta X^3\left(\frac{[Nt]}{N}\right) = 0 \tag{69}$$

over the time range in interval $[Nt]/N \leqslant t < ([Nt]+1)/N$. Displacement of the system on this time interval can be obtained by the same procedure as discussed in Section 2, which gives

$$X_i(t) = e^{-\theta(t-[Nt]/N)}\left[\left(d_i+\frac{\lambda_i}{\omega^2}\right)\cos\left(\xi t-\xi\frac{[Nt]}{N}\right)\right.$$
$$\left.+\frac{1}{\xi}\left(v_i+\theta d_i+\frac{\theta\lambda_i}{\omega^2}\right)\sin\left(\xi t-\xi\frac{[Nt]}{N}\right)-\frac{\lambda_i}{\omega^2}\right] \quad (70)$$

and the corresponding velocity

$$\dot{X}_i(t) = -\theta e^{-\theta(t-[Nt]/N)}\left[\left(d_i+\frac{\lambda_i}{\omega^2}\right)\cos\left(\xi t-\xi\frac{[Nt]}{N}\right)+\frac{1}{\xi}\left(v_i+\theta d_i+\frac{\theta\lambda_i}{\omega^2}\right)\sin\left(\xi t-\xi\frac{[Nt]}{N}\right)\right]$$
$$+e^{-\theta(t-[Nt]/N)}\left[-\xi\left(d_i+\frac{\lambda_i}{\omega^2}\right)\sin\left(\xi t-\xi\frac{[Nt]}{N}\right)+\left(v_i+\theta d_i+\frac{\theta\lambda_i}{\omega^2}\right)\cos\left(\xi t-\xi\frac{[Nt]}{N}\right)\right] \quad (71)$$

where

$$\lambda_i = \beta d_i^3, \quad \xi = (\omega^2-\theta^2)^{1/2} \quad \text{and} \quad \omega^2 > \theta^2.$$

The recurrence relations in this case are expressible as

$$d_i = e^{-\theta/N}\left[\left(d_{i-1}+\frac{\lambda_{i-1}}{\omega^2}\right)\cos\left(\frac{\xi}{N}\right)+\frac{1}{\xi}\left(v_{i-1}+\theta d_{i-1}+\frac{\theta\lambda_{i-1}}{\omega^2}\right)\sin\left(\frac{\xi}{N}\right)\right]-\frac{\lambda_{i-1}}{\omega^2} \quad (72)$$

$$v_i = e^{-\theta/N}\left\{v_{i-1}\cos\frac{\xi}{N}-\sin\frac{\xi}{N}\left[\frac{\theta}{\xi}\left(v_{i-1}+\theta d_{i-1}+\frac{\theta\lambda_{i-1}}{\omega^2}\right)+\xi\left(d_{i-1}+\frac{\lambda_{i-1}}{\omega^2}\right)\right]\right\}. \quad (73)$$

With the same initial conditions as in eqn (6) and conditions of continuity described in eqn (11), a general solution of the problem is obtained in the following form on the entire range $t \geqslant 0$:

$$X(t) = e^{-\theta(t-[Nt]/N)}\left[\cos\left(\xi t-\xi\frac{[Nt]}{N}\right)+\frac{\theta}{\xi}\sin\left(\xi t-\xi\frac{[Nt]}{N}\right)\frac{1}{\xi}\sin\left(\xi t-\xi\frac{[Nt]}{N}t\right)\right]\mathbf{M}$$
$$+\frac{\lambda_i}{\omega^2}e^{-\theta/N(t-[Nt]/N)}\left[\cos\left(\xi t-\xi\frac{[Nt]}{N}\right)+\frac{\theta}{\xi}\sin\left(\xi t-\xi\frac{[Nt]}{N}\right)-e^{\theta/N(t-[Nt]/N)}\right] \quad (74)$$

where the matrix $\mathbf{M}$ is

$$\mathbf{M} = e^{-\theta[Nt]/N}\mathbf{T}^{[Nt]}\begin{bmatrix} d_0 \\ v_0 \end{bmatrix}+\sum_{j=1}^{[Nt]}e^{-\theta j/N}\mathbf{T}^{j-1}\begin{bmatrix} \cos\frac{\xi}{N}+\frac{\theta}{\xi}\sin\frac{\xi}{N}-e^{\theta/N} \\ -\left(\frac{\theta^2}{\xi}+\xi\right)\sin\frac{\xi}{N} \end{bmatrix}\lambda_{[Nt]-j} \quad (75)$$

in which the square matrix

$$\mathbf{T} = \begin{bmatrix} \cos\frac{\xi}{N}+\frac{\theta}{\xi}\sin\frac{\xi}{N} & \frac{1}{\xi}\sin\frac{\xi}{N} \\ -\left(\frac{\theta^2}{\xi}+\xi\right)\sin\frac{\xi}{N} & \cos\frac{\xi}{N}-\frac{\theta}{\xi}\sin\frac{\xi}{N} \end{bmatrix}. \quad (76)$$

Although eqn (74) is a complete solution to eqn (69), eqns (70) and (71) with the recurrence relations (72) and (73) are practical and manageable when used in a computer program to procure a numerical solution.

According to what has been discussed in Section 2, piecewise constant argument $[Nt]/N$ tends to be the continuous time $t$ when $N$ approaches infinity. Hence, as $N \to \infty$, eqn (69) tends to be a continuous Duffing's eqn (68), and theoretically, the solution in the form of eqn (74) will be an accurate solution of the Duffing's eqn (68). The present method may therefore be employed to analyze and numerically solve the nonlinear oscillation problems, in addition to solving the linear and nonlinear oscillations subjected to piecewise constant forces. In fact, if the value of the integer $N$ is properly chosen and large enough, the numerical solution given by eqn (74) will be sufficiently close to the solution of Duffing's eqn (68).

It is significant to note that the solutions given by the present method are continuous in the time interval $[Nt]/N \leqslant t < ([Nt]+1)/N$ and the entire range of time $t \in [0, \infty)$. The time $t$ in a solution may be given any real value, and for any value of $t$ there is a definite $X$ value corresponding to it. In contrast to the present method, the most existing numerical methods such as the average acceleration method, linear acceleration method demonstrated by Weaver *et al.* (1990) and Newmark (1959), Euler's method and Runge–Kutta method presented by Philips *et al.* (1986) and Gerald *et al.* (1989), provide only the solutions at the discrete points, $t_n$, $n = 1, 2, 3, \ldots$. The information in between $t_n$ and $t_{n+1}$ is not available. From the discussion above, it can also be seen that all the terms in the governing equations, except only one term which is piecewise-constant, remain unchanged during the derivation and calculation of the numerical solutions. For instance, the approximate solution of eqn (68) is obtained on the basis of the governing eqn (69) with a piecewise constant argument $\beta X^3([Nt]/N)$. Among the four terms in eqn (69), the first three terms are identical to the first three terms in eqn (68). The information embedded in these three terms is therefore retained in the approximate solution expressed by eqn (74). When the chosen value of $N$ is sufficiently large, $\beta X^3([Nt]/N)$ can be very close to $\beta x^3(t)$, and eqn (74) will then be a good approximation to the solution of eqn (68). Similarly, by the original information carried by the unchanged terms in the equations of motion, the continuous solutions such as those shown in eqns (59), (63), and (70) can be considered as good approximations to the exact solutions of the corresponding continuous systems if a sufficiently large $N$ is chosen.

Employing the methods of average-acceleration and linear-acceleration, Weaver (1990) reported the numerical solutions for the governing eqn (68), but without damping. Under the same conditions, the present method is employed to solve the same governing equation for a comparison. The numerical solution produced by using eqns (70) and (71) with recurrence relations (72) and (73) shows good convergency and matches well with the numerical results obtained by Weaver (1990). Figure 2 gives the numerical results for different values of $N$. It may be noted in the figure that the differences between the numerical solutions decrease as $N$ increases.

(*b*) *Motion of a spring-mass system in the presence of a nonlinear damping and friction.* In solving a nonlinear engineering problem in dynamics, according to the discussion above, governing equation in vibration problems can be linearized on an arbitrary time interval $[Nt]/N \leqslant t < ([Nt]+1)/N$ by replacing a nonlinear term expressed in the form $f(x(t), \dot{x}(t), t)$ with the corresponding piecewise-constant term in the form $g(x([Nt]/N), \dot{x}([Nt]/N), [Nt]/N)$. In carrying out the piecewise constant procedure for a governing equation having more than one non-linear term, two or more terms or variables in the equation may be set in a piecewise-constant form on a chosen time interval to make the resulting equation solvable by the existing methods for solving linear or nonlinear differential equations. As an example, consider the following equation of motion which represents the oscillations of a particle attached to a spring under the influence of friction and damping:

$$\ddot{x}(t) + \omega^2 x(t) + \mu_1 \dot{x}(t) + \mu_2 \dot{x}(t)|\dot{x}(t)| + \mu_3 \dot{x}^3(t) = 0 \tag{77}$$

where $\mu_1$, $\mu_2$ and $\mu_3$ are constants. This equation may be expressed in piecewise-constant form in the interval $[Nt]/N \leqslant t < ([Nt]+1)/N$ as

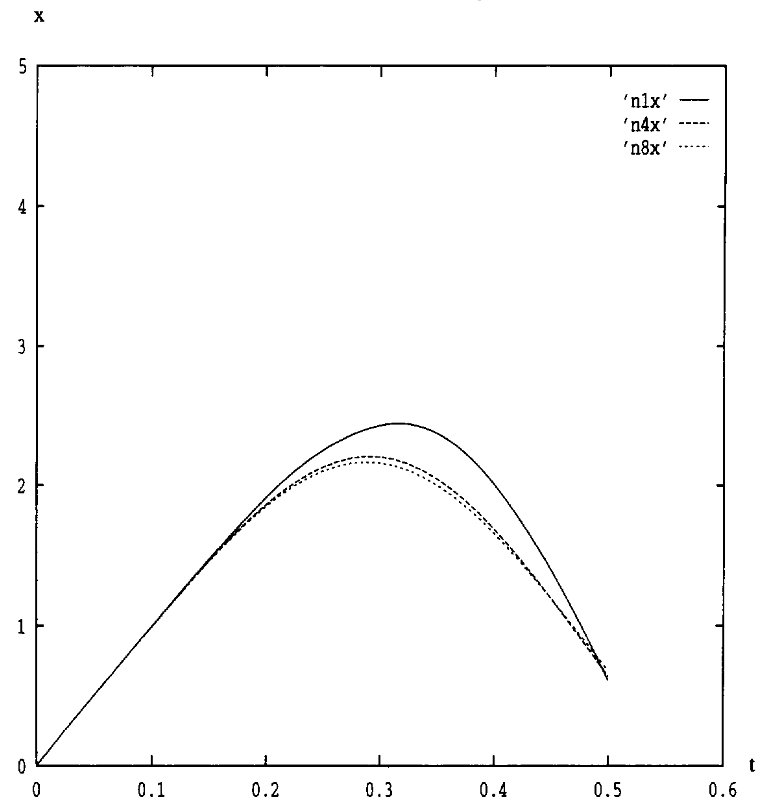


Fig. 2. Convergence of the solution of $m\ddot{x}+k\{x+ax([Nt]/N)^3\} = 0$. $m = 100, k = 400, a = 2, d_0 = 0, v_0 = 10$. '$n1x$' : $N = 10$; '$n4x$' : $N = 40$; '$n8x$' : $N = 80$.

$$\ddot{X}_i(t)+\omega^2 X_i(t)+\mu_1 \dot{X}_i(t)+\mu_2 \dot{X}_i\left(\frac{[Nt]}{N}\right)\left|\dot{X}_i\left(\frac{[Nt]}{N}\right)\right|+\mu_3 \dot{X}_i^3\left(\frac{[Nt]}{N}\right) = 0. \quad (78)$$

The numerical solution of the system (78) together with the recurrence relations can be produced through a procedure similar to that used for the system described by eqn (69). The displacement, velocity and the recurrence relations for numerical calculations are derived as

$$X_i = e^{-\mu_1(t-[Nt]/N)/2}\left\{\left(d_i-\frac{F_i}{\omega^2}\right)\cos\left[\xi\left(t-\frac{[Nt]}{N}\right)\right] + \frac{1}{\xi}\left[v_i+\frac{\mu_1}{2}\left(d_i-\frac{F_i}{\omega^2}\right)\right]\sin\left[\xi\left(t-\frac{[Nt]}{N}\right)\right]\right\}+\frac{F_i}{\omega^2} \quad (79)$$

$$\dot{X}_i = -\frac{\mu_1}{2}e^{-\mu_1(t-[Nt]/N)/2}\left\{\left(d_i-\frac{F_i}{\omega^2}\right)\cos\left[\xi\left(t-\frac{[Nt]}{N}\right)\right] + \frac{1}{\xi}\left[v_i+\frac{\mu_1}{2}\left(d_i-\frac{F_i}{\omega^2}\right)\right]\sin\left[\xi\left(t-\frac{[Nt]}{N}\right)\right]\right\}+e^{-\mu_1(t-[Nt]/N)/2}$$
$$\times\left\{-\left(d_i-\frac{F_i}{\omega^2}\right)\xi\sin\left[\xi\left(t-\frac{[Nt]}{N}\right)\right]+\left[v_i+\frac{\mu_1}{2}\left(d_i-\frac{F_i}{\omega^2}\right)\right]\cos\left[\xi\left(t-\frac{[Nt]}{N}\right)\right]\right\} \quad (80)$$

$$d_i = -e^{-\mu_1/2N}\left\{\left(d_{i-1}-\frac{F_{i-1}}{\omega^2}\right)\cos\frac{\xi}{N}+\frac{1}{\xi}\left[v_{i-1}+\frac{\mu_1}{2}\left(d_{i-1}-\frac{F_{i-1}}{\omega^2}\right)\right]\sin\frac{\xi}{N}\right\}+\frac{F_{i-1}}{\omega^2} \quad (81)$$

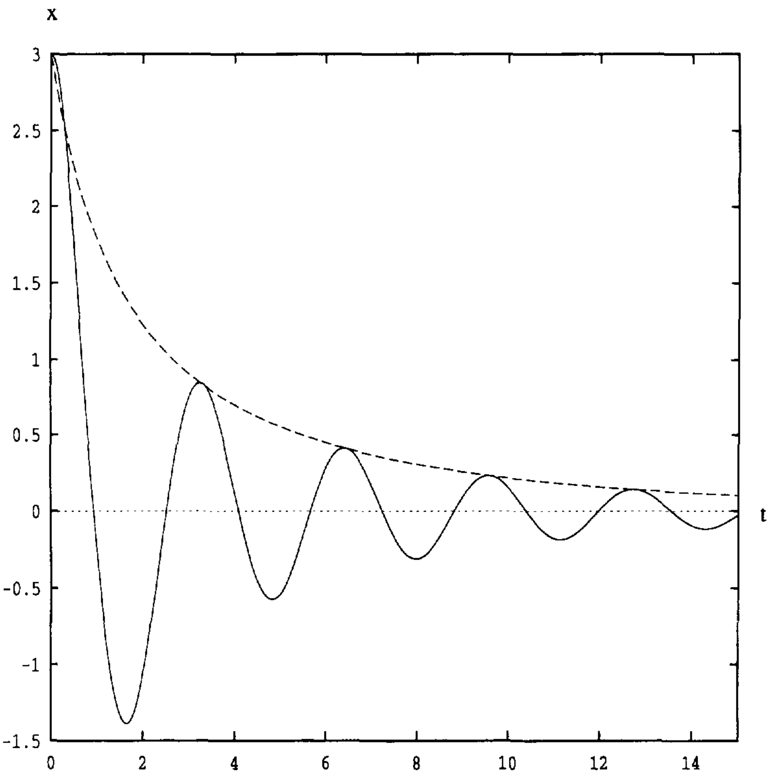


Fig. 3. Numerical solution of $\ddot{x}+\omega^2\dot{x}+\mu_1 x+\mu_2\dot{x}|\dot{x}|+\mu_3\dot{x}^3 = 0$ (solid line) and the first approximation (dashed line). $\omega = 0.5$, $\mu_1 = 4.0$, $\mu_2 = 0.2$, $\mu_3 = 0.0$, $d_0 = 3.0$, $v_0 = 0.0$.

$$v_i = e^{-\mu_1/2N}\left\{v_{i-1}\cos\frac{\xi}{N}-\left[\frac{\mu_1}{2}v_{i-1}+\left(d_{i-1}-\frac{F_{i-1}}{\omega^2}\right)\left(\frac{\mu_1^2}{4\xi}+\xi\right)\right]\sin\frac{\xi}{N}\right\} \tag{82}$$

where

$$\omega^2 > \mu_1^2/4, \quad F_i = -\mu_2 v_i|v_i|-\mu_3 v_i^3, \quad \xi^2 = \omega^2-\frac{\mu_1^2}{4}. \tag{83a,b,c}$$

A numerical solution for this system is illustrated in Fig. 3 with a comparison between the numerical solution given by the present technique and the first approximation for the amplitude of motion given by Baum (1972).

(3) *Chaotic behavior of numerical solutions for nonlinear systems*

In analyzing the nonlinear systems with chaotic behavior, the numerical solutions are known to be very sensitive to initial conditions and time integral step used for numerical computations as discussed by Gwinn *et al.* (1986). In solving the chaotic problems, the present method can still be employed for the purpose of analysis and numerical computation.

In nonlinear and chaotic cases, we also found that the piecewise-constant method and the corresponding Runge–Kutta method give comparatively very close results.

(*a*) *Forced motions of a nonlinear pendulum*. Consider the following equation of motion which may lead to chaos:

$$\ddot{x}+b_1\dot{x}+b_2\sin x = b_3\cos b_4 t \tag{84}$$

where $b_1$, $b_2$, $b_3$ and $b_4$ are parameters of the problem. The system governed by the above equation represents a nonlinear, damped, sinusoidally driven pendulum. Express this equation in a piecewise constant form in interval $[Nt]/N \leqslant t < ([Nt]+1)/N$ as

$$\ddot{X}+b_1\dot{X}+b_2 \sin X\left(\frac{[Nt]}{N}\right) = b_3 \cos b_4 t. \tag{85}$$

The displacement and velocity for the system governed by eqn (85) can be derived through a procedure similar to that used for obtaining eqns (70) and (71) as

$$X_i = A_i+B_i e^{-b_1(t-[Nt]/N)} - \frac{b_3}{b_4^2+b_1^2}\cos b_4 t+\frac{b_1 b_3}{b_4^3+b_1^2 b_4}\sin b_4 t-\frac{b_2}{b_1}\left(t-\frac{[Nt]}{N}\right)\sin d_i \tag{86}$$

$$\dot{X}_i = -b_1 B_i e^{-b_1(t-[Nt]/N)}+\frac{b_3 b_4}{b_4^2+b_1^2}\sin b_4 t+\frac{b_1 b_3}{b_4^2+b_1^2}\cos b_4 t-\frac{b_2}{b_1}\sin d_i \tag{87}$$

where

$$B_i = \frac{1}{b_1}\left(\frac{b_3 b_4}{b_4^2+b_1^2}\sin b_4\frac{i}{N}+\frac{b_1 b_3}{b_4^2+b_1^2}\cos b_4\frac{i}{N}-\frac{b_2}{b_1}\sin d_i - v_i\right) \tag{88}$$

$$A_i = d_i - B_i + \frac{b_3}{b_4^2+b_1^2}\cos b_4\frac{i}{N}-\frac{b_1 b_3}{b_4^3+b_1^2 b_4}\sin b_4\frac{i}{N}. \tag{89}$$

The corresponding recurrence relations are

$$d_i = A_{i-1}+B_{i-1}e^{-b_1/N}-\frac{b_3}{b_4^2+b_1^2}\cos b_4\frac{[Nt]}{N}+\frac{b_3 b_1}{b_4^3+b_1^2 b_4}\sin b_4\frac{[Nt]}{N}-\frac{b_2}{b_1 N}\sin d_{i-1} \tag{90}$$

$$v_i = -b_1 B_{i-1}e^{-b_1/N}+\frac{b_3 b_4}{b_4^2+b_1^2}\sin b_4\frac{[Nt]}{N}+\frac{b_1 b_3}{b_4^2+b_1^2}\cos b_4\frac{[Nt]}{N}-\frac{b_2}{b_1}\sin d_{i-1}. \tag{91}$$

Making use of the data generated by employing solutions (86) and (87) together with the recurrence relations (90) and (91), the numerical results are graphically presented in Figs 4 and 5. Figure 4 illustrates the phase trajectory and Fig. 5 shows the corresponding Poincare map of a chaotic case for the motion governed by equation (84). Under the same conditions, Figs 4 and 5 are almost identical to the phase trajectory and the Poincare map provided by Gwinn (1986), Blackburn *et al.* (1989) and Baker *et al.* (1990).

(*b*) *Duffing's equation with periodic excitation.* A Duffing's equation in the following form was investigated by Ueda (1980)

$$\ddot{x}+c_1\dot{x}+c_2 x^3 = c_3 \cos c_4 t \tag{92}$$

where $c_1$, $c_2$, $c_3$, $c_4$ are constants. In order to solve this nonlinear equation by a piecewise

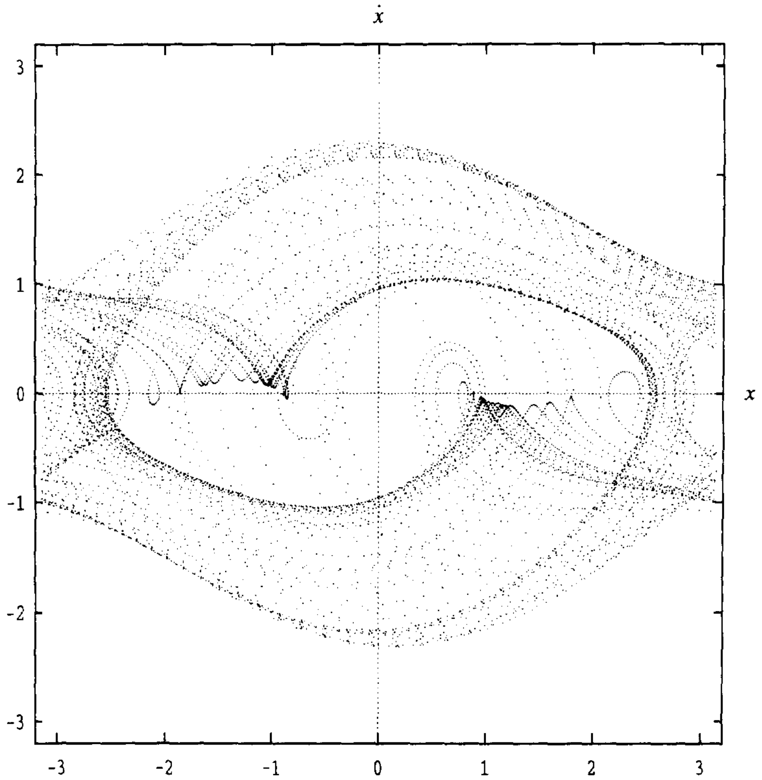


Fig. 4. Phase trajectory of a steady state solution for $\ddot{x}+b_1\dot{x}+b_2\sin x = b_3\cos b_4 t$. $b_1 = 0.5$, $b_2 = 1.0$, $b_3 = 1.15$, $b_4 = 2/3$, $d_0 = -2.5$, $v_0 = 0$.

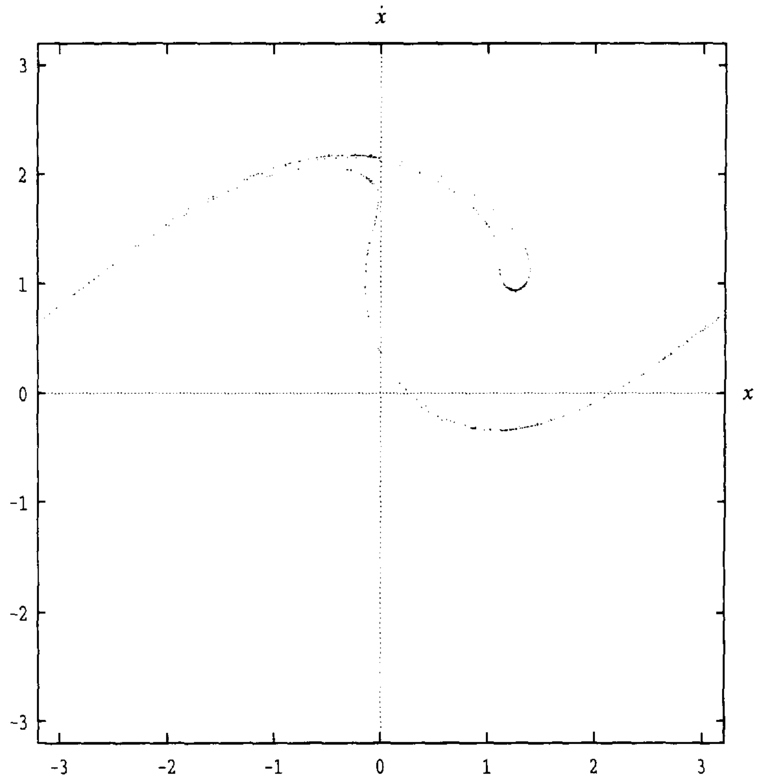


Fig. 5. Poincare map corresponding to the phase trajectory in Fig. 4.

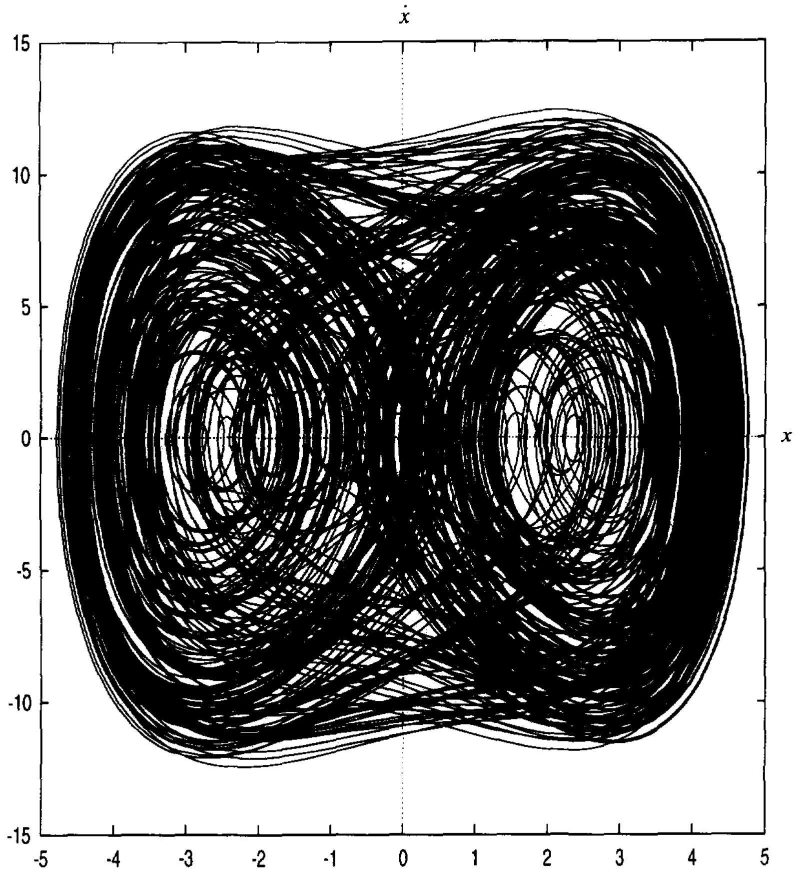


Fig. 6. Phase trajectory corresponding to the motion governed by $\ddot{x}+c_1\dot{x}+c_2x^3 = c_3\cos c_4t$. $c_1 = 0.05$, $c_2 = 1.0$, $c_3 = 23.9$, $c_4 = 1.0$.

constant procedure, eqn (92) is expressed as

$$\ddot{X}+c_1\dot{X}+c_2X^3\left(\frac{[Nt]}{N}\right) = c_3\cos c_4t. \tag{93}$$

Noting that the third term on the left-hand-side of eqn (93) is a constant in $[Nt]/N \leqslant t < ([Nt]+1)/N$, whereas the other terms are identical to the corresponding terms in eqn (84), approximate or numerical solution of eqn (92) can be derived through the same procedure as demonstrated in part (a) and the displacement, velocity and corresponding recurrence relations are in similar forms as those shown in eqns (86), (87), (90) and (91).

A chaotic case governed by the equation of motion (92) is examined by using the solution obtained through the piecewise constant procedure and its phase trajectory is shown in Fig. 6 which matches well with the results given by Ueda. Sensitivity of the motion to the initial conditions is found and illustrated in Fig. 7. Under certain conditions the motion corresponding to eqn (92) may be periodic or nonperiodic. Figure 8 exhibits the trajectory of a periodic case and Fig. 9 gives the Poincare map of a nonperiodic motion of the system.

## 5. CONCLUSIONS

The equations of motion given in Sections 2 and 3 are fundamental and of great practical importance in connection with vibration problems. Conventionally, in solving these second-order differential equations in closed form, a certain form of the solution is assumed beforehand with undetermined constants. Using the assumed solution, an algebraic equation known as the characteristic equation is found. The solution of the characteristic equation together with the assumed solution and the initial conditions yields the complete

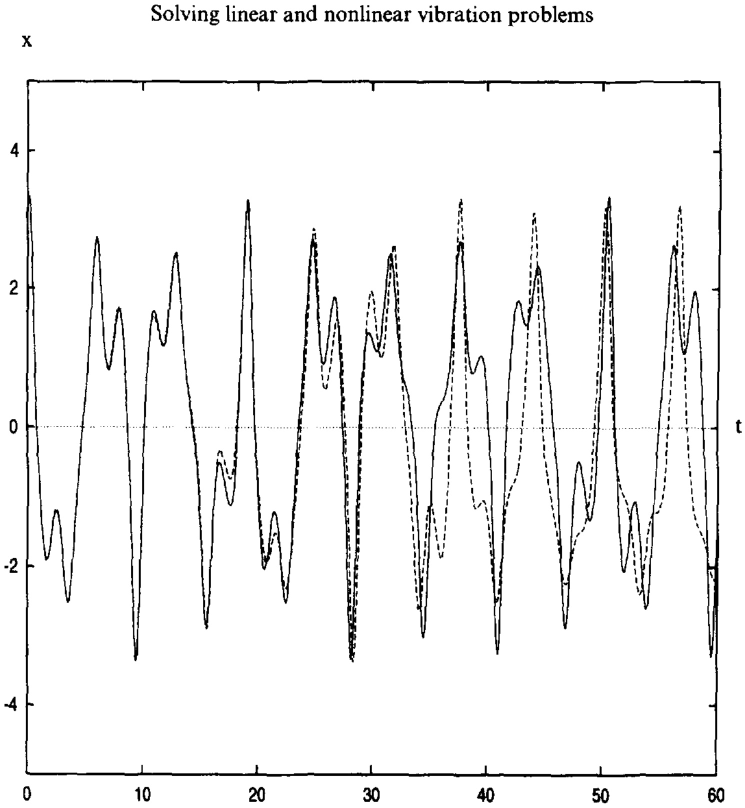


Fig. 7. Demonstration of sensitivity of a nonlinear system governed by $\ddot{x}+c_1\dot{x}+c_2x^3 = c_3\cos c_4t$ to initial conditions. $c_1 = 0.05$, $c_2 = 1.0$, $c_3 = 7.5$, $c_4 = 1.0$. Initial conditions: $d_0 = 3.0$ and $v_0 = 4.0$ (solid line); $d_0 = 3.01$ and $v_0 = 4.01$ (dashed line).

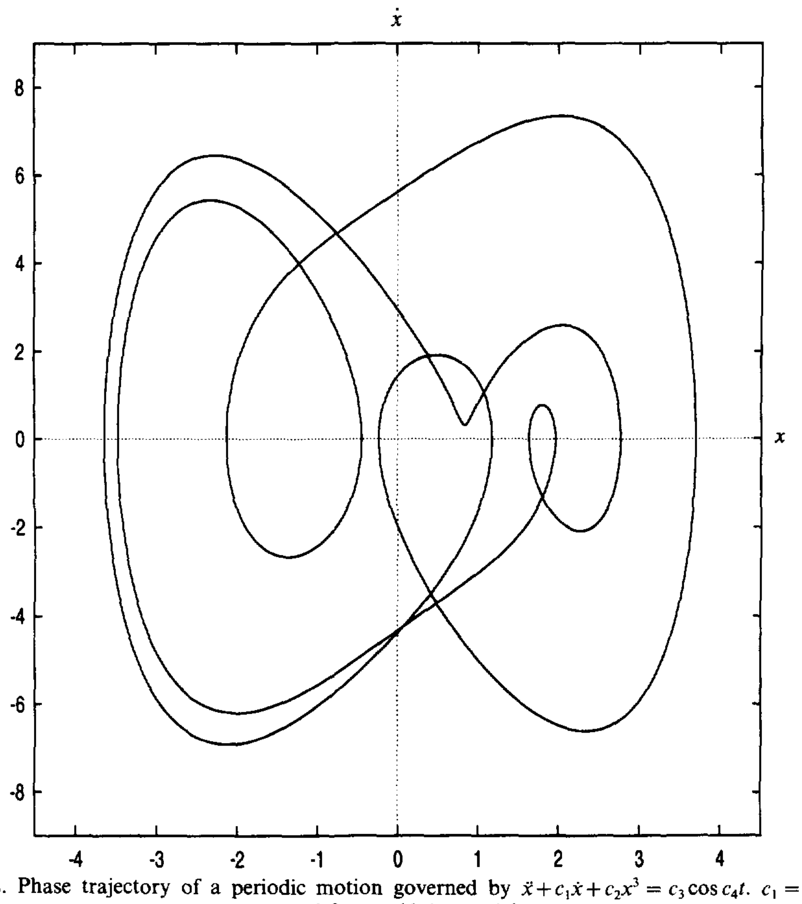


Fig. 8. Phase trajectory of a periodic motion governed by $\ddot{x}+c_1\dot{x}+c_2x^3 = c_3\cos c_4t$. $c_1 = 0.26$, $c_2 = 1.0$, $c_3 = 11.4$, $c_4 = 1.0$.

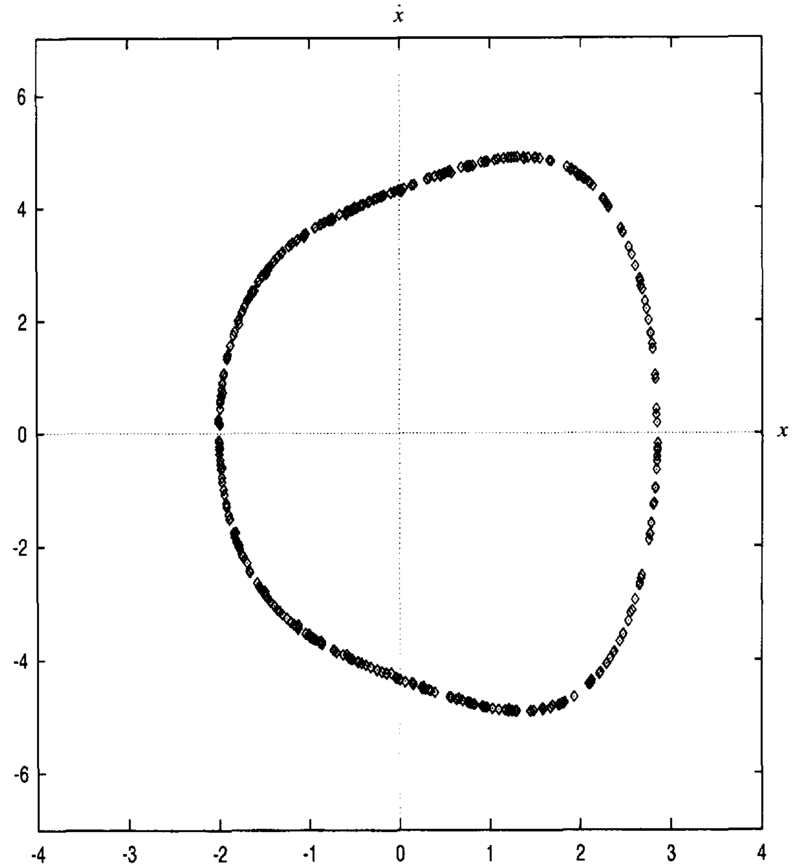

Fig. 9. Poincare map of a nonperiodic motion governed by $\ddot{x}+c_1\dot{x}+c_2x^3 = c_3\cos c_4t$. $c_1 = 0.00001$, $c_2 = 1.0$, $c_3 = 2.0$, $c_4 = 1.0$.

solution in the closed form. However, in solving the second-order differential equations by the present method, the solution is considered on a small time segment $[Nt]/N \leqslant t < ([Nt]+1)/N$ where one of the terms in the differential equation is treated as a constant. Therefore, the differential equation on this time segment is simplified in a way such that the corresponding solution can be easily obtained (direct integration is available for the case (a) in Section 3). If a range from zero to $t$ is considered, $X([Nt]/N)$ is then piecewise constant related to the unknown function $X(t)$. Under the continuity conditions, an approximate solution can be constructed by combining all solutions corresponding to small time segments over the range from zero to $t$. When the parameter $N$ approaches infinity and the length of the time segment tends to zero, the solution corresponding to a given equation of motion converges to its exact solution. In applying the present method for solving vibration problems, there is no need to assume a solution in advance and consequently there is no need to construct a characteristic equation as those demonstrated by Lancaster (1966) and Weaver *et al.* (1990).

In addition to a new method for solving vibration problems governed by eqn (1), with the introduction of the argument $[Nt]/N$, the gap between a continuous system and a corresponding piecewise constant system has been filled.

The piecewise constant method presented in this paper has also been employed in numerically solving linear and nonlinear vibration problems. Numerical experiments demonstrate that the method present is efficient and has good convergency. Formulae for numerical computation for solving various vibratory problems are provided in the present paper. The solutions and the recurrence relations for numerical calculation can be conveniently developed and applied on a computer.

In comparison with the existing numerical methods, there are a few points which need to be stated.

1. In contrast to the existing numerical methods, which give solutions only at discrete points, numerical solution derived by the present method are continuous everywhere along the entire time range considered.
2. In numerically solving the vibration problems, usually, the second-order differential equation has to be transformed into a system of first-order differential equations. Numerical solutions corresponding to the first-order differential equations are then developed by employing the mathematical operations such as the linearization discussed by Smith *et al.* (1985) or Taylor expansion demonstrated by Dahlquist *et al.* (1974). In doing so, the clarity of the physical meaning involved in the original equation of motion is quite often blurred in the manipulations of the pure mathematical expressions. However, the present method tries to keep more physical information in the original governing equation unchanged. In each time interval considered, there is a vibration system corresponding to it. Due to the maintenance of the original physical information, the continuous solution given by the present method is a good approximation to the exact or accurate solution in the time interval and along the entire time range. Theoretically, the approximate solutions produced by the present method become the exact solutions to the dynamical systems as $N$ approaches infinity.
3. Iteration is a major operation for the numerical calculations of many numerical methods. When the local initial conditions are given, the iteration must be repeatedly carried out to obtain the numerical solution at the end of the time interval. Nevertheless, there is no iteration involved in the numerical calculations by using the present method. Once the local initial conditions are available, the continuous solution of the time interval, including the end of the interval, can be directly obtained.

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